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EVERSION OF A FOLD MAP OF S^2 TO $\mathbf{R}^2$ WITH ONE SINGULAR SET

MINORU YAMAMOTO (山本 稔; (前) 九州大学, (現) 北海道大学)
(OLD ADDRESS) GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY
(CURRENT ADDRESS) DEPARTMENT OF MATHEMATICS, HOKKAIDO UNIVERSITY

1. Introduction

In the following, all manifolds and maps are differentiable of class C^∞.

Let M be an n-dimensional closed manifold, N an n-dimensional manifold and $f : M \to N$ a map of M into N. We denote by $S(f)$ the set of the points in M where the rank of the differential of f is strictly less than n. We call $S(f)$ the *singular set* of f and $f(S(f))$ the *singular value set* of f. We say that a map $f : M \to N$ is a *fold map* if there exist local coordinate systems $(x_1, x_2, \ldots, x_n)$ around $q \in M$ and $(y_1, y_2, \ldots, y_n)$ around $f(q) \in N$ such that f has one of the following forms:

$$(y_1 \circ f, y_2 \circ f, \ldots, y_{n-1} \circ f, y_n \circ f) = \begin{cases} (x_1, x_2, \ldots, x_{n-1}, x_n), & q\text{: regular point,} \\ (x_1, x_2, \ldots, x_{n-1}, x_n^2), & q\text{: fold point.} \end{cases}$$

Note that for a fold map $f : M \to N$, $S(f)$ is an $(n-1)$-dimensional submanifold of M. If the restricted map $f|S(f) : S(f) \looparrowright N$ is an immersion with normal crossings, we call f a *stable fold map*.

Let V be an $(n-1)$-dimensional submanifold of M and $f : M \to N$ a fold map such that $S(f) = V$. We denote by $\mathcal{F}(M, N; V)$ the set of such fold maps. Note that $\mathcal{F}(M, N; V)$ is the subspace of $C^\infty(M, N)$ having the Whitney C^∞-topology. Let T be a tubular neighborhood of V in M such that there exists a fiber involution of it, $h : T \to T$, whose fixed points set is V and the composition $(f|T) \circ h$ coincides with $f|T$. Note that for any $f \in \mathcal{F}(M, N; V)$, we may assume that T does not depend on f but depends on M and V. For $\widetilde{M} = \mathrm{cl}(M \setminus T)$, the closure of $M \setminus T$, $f|\widetilde{M} : \widetilde{M} \to N$ is an immersion.

In [1, 2], Eliashberg studied the existence of a fold map $f : M \to N$. In the appendix of [2], he proved that the number of the connected components of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ is strictly four, where S^2 is an oriented 2-dimensional sphere, $\mathbf{R}^2$ is the oriented plane and S^1_0 is the equator of S^2. We denote by $\mathcal{S}_1, \mathcal{S}_2, \mathcal{E}_1$ and $\mathcal{E}_2$ the connected components of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. We call a fold map f in $\mathcal{S}_i$ a *standard* fold map and in $\mathcal{E}_i$ an *exotic* fold map $(i = 1, 2)$. In the same paper, he showed the representative elements of each connected components of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. Let $e : S^2 \to \mathbf{R}^2$ be the representative element of $\mathcal{E}_1$ such that Eliashberg gave this map in [2] ([1]). This fold map is constructed by using two immersed disks called *Milnor's examples*. We can construct

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another exotic fold map $\tilde{e} \in \mathcal{E}_1$ by using these two Milnor's examples. Then, there exists a homotopy $E : S^2 \times [-1, 1] \to \mathbf{R}^2$ such that $e_{-1} = e$, $e_1 = \tilde{e}$ and $e_t \in \mathcal{E}_1$, where e_t is defined by $e_t(x) = E(x, t)$ ($x \in S^2, t \in [-1, 1]$). We call such a homotopy a *fold eversion* between e and $\tilde{e}$.

In [2], Eliashberg only stated the existence of a fold eversion. As the theorem of sphere eversion [7], it is difficult to give a fold eversion at first glance. In this report, we construct a fold eversion between e and $\tilde{e}$ concretely as Morin and Petit constructed a sphere eversion concretely (see [6]).

The report is organized as follows.

In Section 2, we characterize each connected component of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ and construct fold maps e and $\tilde{e}$. We observe local behaviors of a homotopy of fold maps.

In Section 3, we give a fold eversion between e and $\tilde{e}$ concretely.

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2. Preliminaries

In this section, we state each connected component of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ precisely. We also see local behaviors of a homotopy of fold maps.

Let S^2 be an oriented 2-dimensional sphere, $\mathbf{R}^2$ the oriented plane and S^1_0 the equator of S^2. Let T be a tubular neighborhood of S^1_0 in S^2 and we fix a trivialization $T \cong S^1_0 \times [-1, 1]$ such that $S^1_0 = S^1_0 \times \{0\}$. We have a fiber involution $h : S^1_0 \times [-1, 1] \to S^1_0 \times [-1, 1]$ such that $h(x, t) = (x, -t)$. Then we may assume that for any $f \in \mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$, we have

$$(f|S^1_0 \times [-1, 1])(x, t) = (f|S^1_0 \times [-1, 1])(x, -t) \tag{2.1}$$

and

$$f|S^1_0 \times \{t\} \text{ is sufficiently close to } f|S^1_0 \times \{1\}. \tag{2.2}$$

Here, $\mathbf{R}^2$ has the Euclidean metric. We denote by D^2_N and D^2_S each connected component of $\mathrm{cl}(S^2 \setminus T)$.

Definition 2.1. Let $f : S^2 \to \mathbf{R}^2$ be a fold map in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. We say that $f|D^2_N$ and $f|D^2_S$ are the *same extensions* of $f|\partial T$ if there exists an orientation reversing diffeomorphism $k : D^2_N \to D^2_S$ such that $k|\partial D^2_N = h$ and $f|D^2_S \circ k = f|D^2_N$. Otherwise, we say that $f|D^2_N$ and $f|D^2_S$ are the *different extensions* of $f|\partial T$.

Then, Eliashberg's theorem is stated as follows.

Theorem 2.2 (Eliashberg [2]). *Each connected component of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0) = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ consists of all fold maps satisfying the following properties.*

(1) *The connected component $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) consists of all fold maps $f : S^2 \to \mathbf{R}^2$ in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ such that $f|D^2_N$ and $f|D^2_S$ are the same extensions of $f|\partial T$. We set the*

orientation of S^2 so that $f|D^2_N$ is the orientation preserving (resp. reversing) immersion and $f|D^2_S$ is the orientation reversing (resp. preserving) immersion.

(2) *The connected component $\mathcal{E}_1$ (resp. $\mathcal{E}_2$) consists of all fold maps $f : S^2 \to \mathbf{R}^2$ in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ such that $f|D^2_N$ and $f|D^2_S$ are the different extensions of $f|\partial T$. We set the orientation of S^2 so that $f|D^2_N$ is the orientation preserving (resp. reversing) immersion and $f|D^2_S$ is the orientation reversing (resp. preserving) immersion.*

Let $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ be the canonical projection defined by $\pi(x_1, x_2, x_3) = (x_1, x_2)$ and $i : S^2 \to \mathbf{R}^3$ the inclusion defined by $i(S^2) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 = 1\}$ and $i(S^1_0) = \{(x_1, x_2, x_3) \in i(S^2) \mid x_3 = 0\}$. If we choose a suitable orientation on S^2, $s = \pi \circ i$ is a stable fold map in $\mathcal{S}_1$. The images of $s(D^2_N)$ and $s(D^2_S)$ are depicted as in FIGURE 1.

FIGURE 1

In [2] ([1]), Eliashberg gave a representative element, $e : S^2 \to \mathbf{R}^2$, of $\mathcal{E}_1$. Let D^2 be an oriented 2-dimensional disk. Let m_1 and $m_2 : D^2 \looparrowright \mathbf{R}^2$ be two orientation preserving immersions called *Milnor's examples* (see FIGURE 2). Note that m_1 and m_2 are the different extensions of $m_1|\partial D^2 = m_2|\partial D^2$.

FIGURE 2

Let $g_N : D^2_N \to D^2$ be an orientation preserving diffeomorphism and $g_S : D^2_S \to D^2$ an orientation reversing diffeomorphism such that $g_S \circ h|\partial D^2_N = g_N|\partial D^2_N$ holds. Then, we have the desired fold map $e \in \mathcal{E}_1$ such that $e|D^2_N = m_1 \circ g_N$, $e|D^2_S = m_2 \circ g_S$ and $e|T$ satisfies the conditions (2.1) and (2.2). The image of $e(D^2_N)$ is depicted as in FIGURE 3 (a) and $e(D^2_S)$ is depicted as in FIGURE 3 (b).

FIGURE 3

If we exchange these two Milnor's examples on D^2_N and D^2_S, we have another exotic fold map $\widetilde{e} \in \mathcal{E}_1$ such that $\widetilde{e}|D^2_N = m_2 \circ g_N$, $\widetilde{e}|D^2_S = m_1 \circ g_S$ and $\widetilde{e}|T$ satisfies the conditions (2.1) and (2,2). The image of $\widetilde{e}(D^2_N)$ is depicted as in FIGURE 4 (a) and $\widetilde{e}(D^2_S)$ is depicted as in FIGURE 4 (b).

FIGURE 4

Note that e and $\widetilde{e}$ are stable fold maps. In FIGURES 3 and 4, gray strips are the image of rectangles properly embedded in D^2_N and D^2_S, respectively. We draw these gray strips so that they help the readers to understand how to extend $e|\partial T$ (resp. $\widetilde{e}|\partial T$) to $e|D^2_N$ and $e|D^2_S$ (resp. $\widetilde{e}|D^2_N$ and $\widetilde{e}|D^2_S$). They also help the readers to understand $e|D^2_N$ and $e|D^2_S$ (resp. $\widetilde{e}|D^2_N$ and $\widetilde{e}|D^2_S$) are the different extensions of $e|\partial T$ (resp. $\widetilde{e}|\partial T$).

Let f and g be fold maps in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ such that f is a stable fold map. Let $y_g \in g(S(g))$ be a singular value of g. Suppose that there exists a singular value $y_f \in f(S(f))$ such that a map germ $g : (S^2, g^{-1}(y_g) \cap S(g)) \to (\mathbf{R}^2, y_g)$ is $\mathcal{A}$-equivalent to a map germ $f : (S^2, f^{-1}(y_f) \cap S(f)) \to (\mathbf{R}^2, y_f)$. Then, we call y_g a *stable fold singular value* of g.

Let f and $g : S^2 \to \mathbf{R}^2$ be two stable fold maps such that they are in the same connected component of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. By the relative version of the parameterized multi-transversality theorem, there exists a homotopy $F : S^2 \times [-1,1] \to \mathbf{R}^2$ such that F satisfies the following properties.

(1) For any $t \in [-1,1]$, $f_t : S^2 \to \mathbf{R}^2$ is a fold map such that $f_{-1} = f$, $f_1 = g$ and f_t and f are in the same connected component of $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. Here, f_t is defined by $f_t(x) = F(x,t)$.

(2) There is a finite set of parameter values $-1 < t_1 < t_2 < \cdots < t_l < 1$ (possibly empty) in the open interval $(-1,1)$ such that the following conditions hold.

(2-1) For any $t \in [-1,1] \setminus \{t_1, \ldots, t_l\}$, $f_t : M \to \mathbf{R}^2$ is a stable fold map.

(2-2) For each t_i $(i = 1, \ldots, l)$, f_{t_i} has the unique singular value $y_i \in f_{t_i}(S(f_{t_i}))$ which is not stable fold singular value of f_{t_i} $(i = 1, \ldots, l)$. The map germ

$$F : (S^2 \times (t_i - \varepsilon, t_i + \varepsilon), (f_{t_i}^{-1}(y_i) \cap S^1_0) \times \{t_i\}) \to (\mathbf{R}^2, y_i)$$

is $\mathcal{A}$-equivalent to one of the 1-parameter unfoldings in TABLE 1, where ε is a sufficiently small positive real number.

We call such an F a *generic fold homotopy* between f and g. We say that each t_i a *codimension* 1 *bifurcation value* of F and each f_{t_i} a *codimension* 1 *fold map* in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$ $(i = 1, \ldots, l)$. We say that f is the *initial stable fold map* of F and g is the *terminal stable fold map* of F. We denote by Γ_1 the set of all codimension 1 fold maps in $\mathcal{F}(S^2, \mathbf{R}^2; S^1_0)$. By using local normal forms in TABLE 1, Γ_1 is classified into five strata, $J^\star_*$ and T_* $(\star = +, -$ and $* = 1, 2)$. Note that each stratum may not necessarily be connected.

Remark 2.3. The relative multi-transversality theorem is stated and proved in [4] and the parameterized relative multi-transversality theorem is stated in [8]. The $\mathcal{A}$-equivalence classification of map germs $g : (\mathbf{R}^2, S) \to (\mathbf{R}^2, 0)$ and their 1-parameter unfoldings has been studied by Gibson and Hobbs [3]. Here, S consists of finitely many isolated points of $g^{-1}(0)$.

type	normal form $G(x,y,t)$
J^+	$(x_1, y_1^2 + t), (x_2, x_2^2 + y_2^2)$
J_1^-	$(x_1, -y_1^2 + t), (x_2, x_2^2 + y_2^2)$
J_2^-	$(x_1, y_1^2 + t), (x_2, x_2^2 - y_2^2)$
T_1	$(x_1 + y_1^2, x_1 - y_1^2 + t), (x_2, y_2^2), (-y_3^2, x_3)$
T_2	$(x_1 + y_1^2, x_1 - y_1^2 + t), (x_2, y_2^2), (y_3^2, x_3)$

TABLE 1. 1-parameter unfoldings

Let $G : (\mathbf{R}^2 \times \mathbf{R}, S \times \{0\}) \to (\mathbf{R}^2, 0)$ be a 1-parameter unfolding in TABLE 1. We define $g_t : \mathbf{R}^2 \to \mathbf{R}^2$ by $g_t(x) = G(x,t)$ and suppose that $S \subset S(g_0)$. Using the local normal forms in TABLE 1, we see that the deformations of set germs $g_t(\mathbf{R}^2)$ around $0 \in \mathbf{R}^2$ are as depicted in FIGURE 5.

$\boxed{\text{FIGURE 5}}$

Let $F : S^2 \times [-1,1] \to \mathbf{R}^2$ be a generic fold homotopy such that $0 \in [-1,1]$ is the unique codimension 1 bifurcation value of F. Then we say that F crosses Γ_1 positively at f_0 if one of the following holds.

(1) When $f_0 \in J^+$, J_1^- and J_2^-, the number of normal crossing points of $f_1(S(f_1))$ is greater than that of $f_{-1}(S(f_{-1}))$.

(2) When $f_0 \in T_1$ and T_2, the number of preimage over a point in the new-born triangle of $f_1(S(f_1))$ is greater than that over a point in the vanishing triangle of $f_{-1}(S(f_{-1}))$.

If a generic fold homotopy F does not satisfy the above property, then we say that F crosses Γ_1 negatively at f_0.

3. Fold eversion between e and $\widetilde{e}$

In this section, we concretely construct a fold eversion $E : S^2 \times [-1,1] \to \mathbf{R}^2$ between e and $\widetilde{e}$ such that E is a generic fold homotopy.

To describe a variant of such a fold eversion E, we describe a finite series of images of stable fold map, $e_t(D_N^2)$ and $e_t(D_S^2)$, through which the reader can imagine the smooth fold eversion. Note that for any $t \in [-1,1]$, $e_t|T$ satisfies the conditions (2.1) and (2.2) and e_t is defined by $e_t(x) = E(x,t)$.

The initial stable fold map of E is $e = e_{-1}$ and see FIGURE 6.

FIGURE 6

The stable fold map e_{s_1}, FIGURE 7, is obtained from e by crossing J^+ positively four times.

FIGURE 7

The stable fold map e_{s_2}, FIGURE 8, is obtained from e_{s_1} by crossing J^+, J_1^- and T_1 positively twice, T_2 positively four times and J_2^- negatively twice.

FIGURE 8

The stable fold map e_{s_3}, FIGURE 9, is obtained from e_{s_2} by crossing J^+ and T_1 positively twice and J_2^- negatively once.

FIGURE 9

The stable fold map e_{s_4}, FIGURE 10, is obtained from e_{s_3} by crossing T_2 positively twice.

FIGURE 10

The stable fold map e_{s_5}, FIGURE 11, is obtained from e_{s_4} by crossing J^+ positively twice, T_2 positively four times and J_2^- negatively twice.

FIGURE 11

The stable fold map e_{s_6}, FIGURE 12, is obtained from e_{s_5} with the rotation of $\pi/2$. We see that $e_{s_5}(D_N^2) = e_{s_6}(D_S^2)$ and $e_{s_5}(D_S^2) = e_{s_6}(D_N^2)$ hold if we ignore the orientations on D_N^2 and D_S^2.

FIGURE 12

We obtain the terminal stable fold map, $e_1 = \widetilde{e}$, of E (FIGURE 13) from e_{s_6} by reversing the generic fold homotopy between e and e_{s_5} constructed in FIGURES 6–11

FIGURE 13

Then, we have the desired fold eversion $E : S^2 \times [-1,1] \to \mathbf{R}^2$ between e and $\tilde{e}$. In FIGURES 6–13, we omit the orientations on $e_t(D_N^2), e_t(D_S^2)$ and $\mathbf{R}^2$. Gray strips are the image of rectangles properly embedded in D_N^2 and D_S^2, respectively. We draw the gray strips so that they help the readers to understand how to extend $e_t|\partial T$ to $e_t|D_N^2$ and $e_t|D_S^2$ $(t = \{-1, s_1, s_2, \ldots, s_6, 1\})$.

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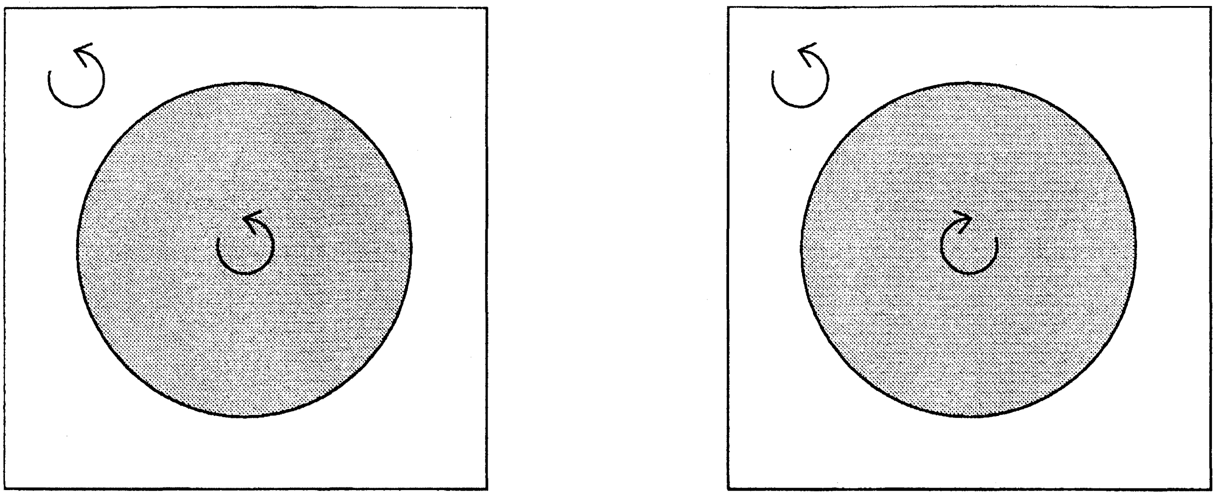
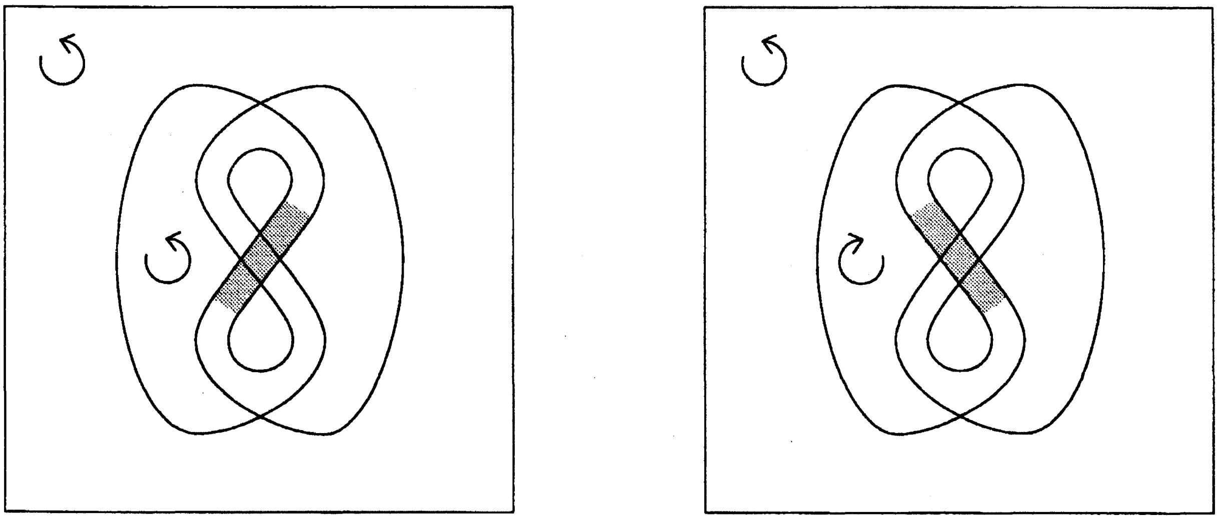
E-mail address: (old) `minomoto@math.kyushu-u.ac.jp`, (current) `minomoto@math.sci.hokudai.ac.jp`



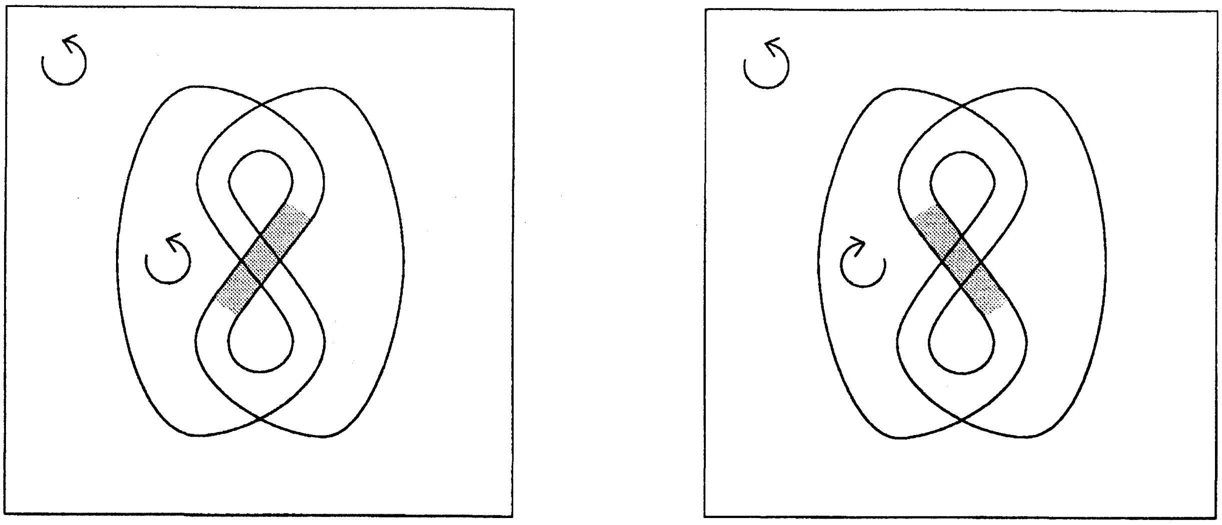
$s(D^2_N)$ $s(D^2_S)$

FIGURE 1. The stable fold map s



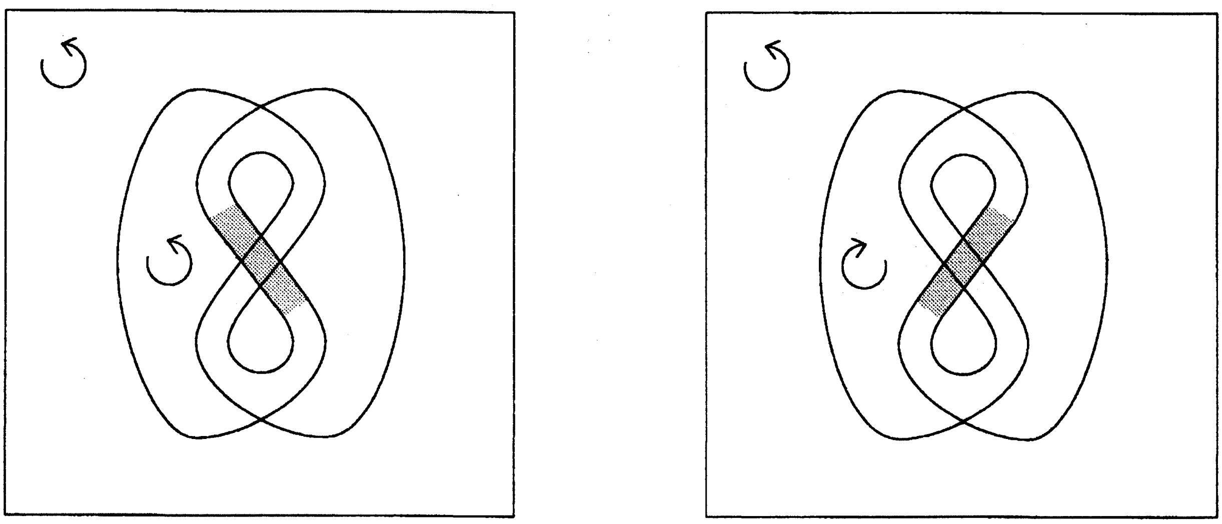
$m_1(D^2)$ $m_2(D^2)$

FIGURE 2. Milnor's examples m_1 and m_2



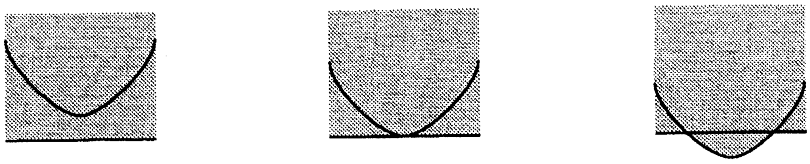
(a) $e(D_N^2)$ (b) $e(D_S^2)$

FIGURE 3. The stable fold map e



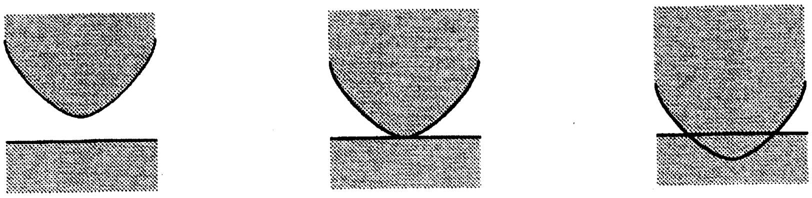
(a) $\widetilde{e}(D_N^2)$ (b) $\widetilde{e}(D_S^2)$

FIGURE 4. The stable fold map $\widetilde{e}$



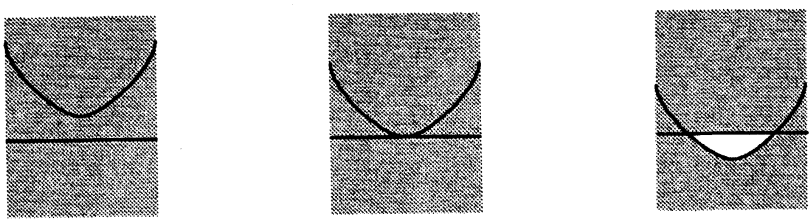
g_{-1} g_0 g_1

(1) if 0 corresponds to J^+



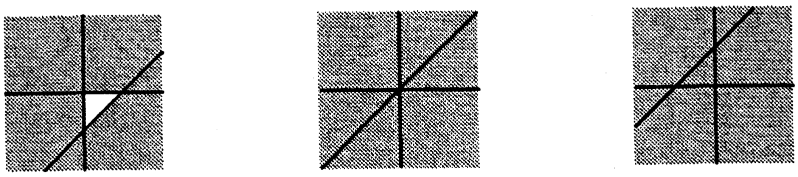
g_{-1} g_0 g_1

(2) if 0 corresponds to J_1^-



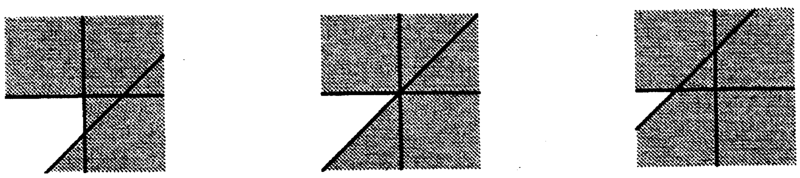
g_{-1} g_0 g_1

(3) if 0 corresponds to J_2^-



g_{-1} g_0 g_1

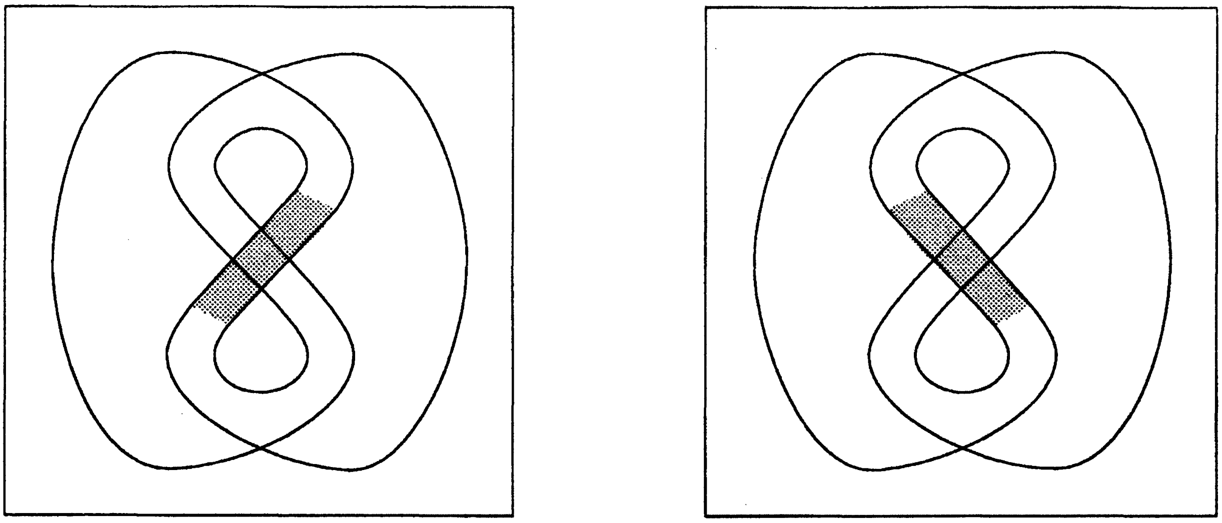
(4) if 0 corresponds to T_1



g_{-1} g_0 g_1

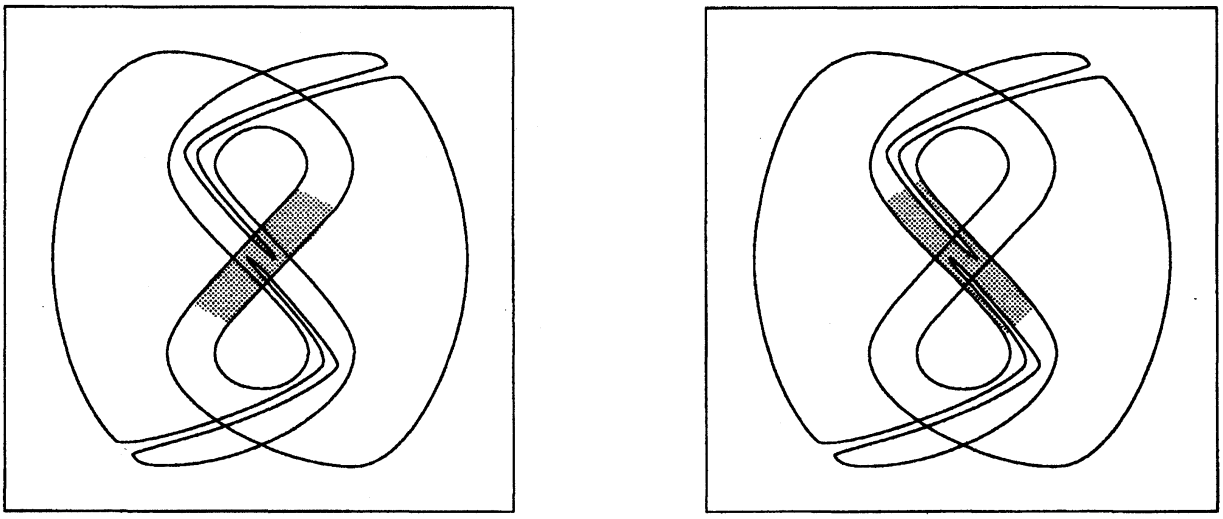
(5) if 0 corresponds to T_2

FIGURE 5



$e(D^2_N)$ $e(D^2_S)$

FIGURE 6. The stable fold map e



$e_{s_1}(D^2_N)$ $e_{s_1}(D^2_S)$

FIGURE 7. The stable fold map e_{s_1}

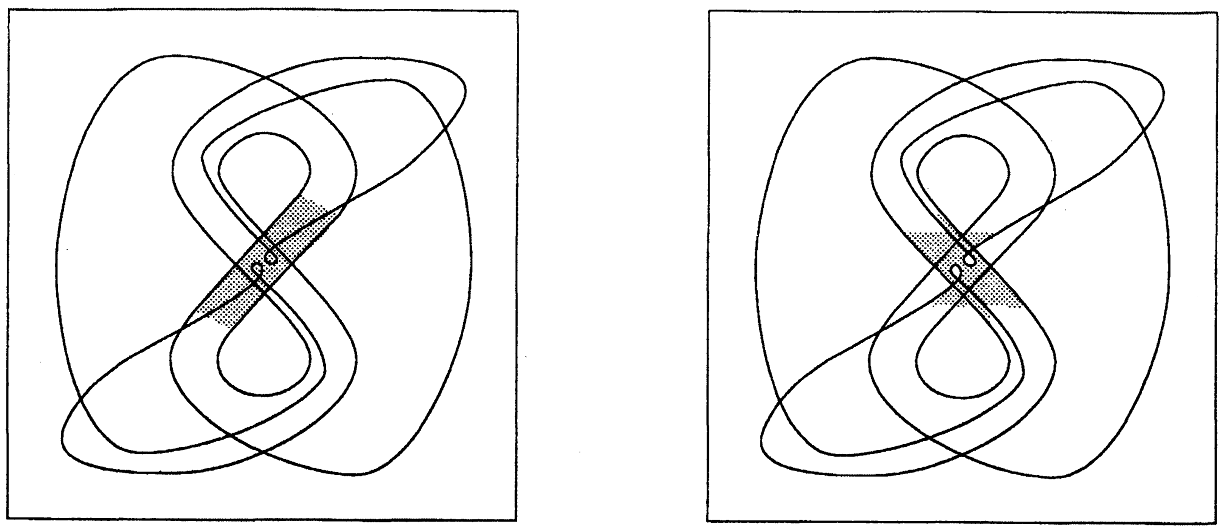


FIGURE 8. The stable fold map e_{s_2}

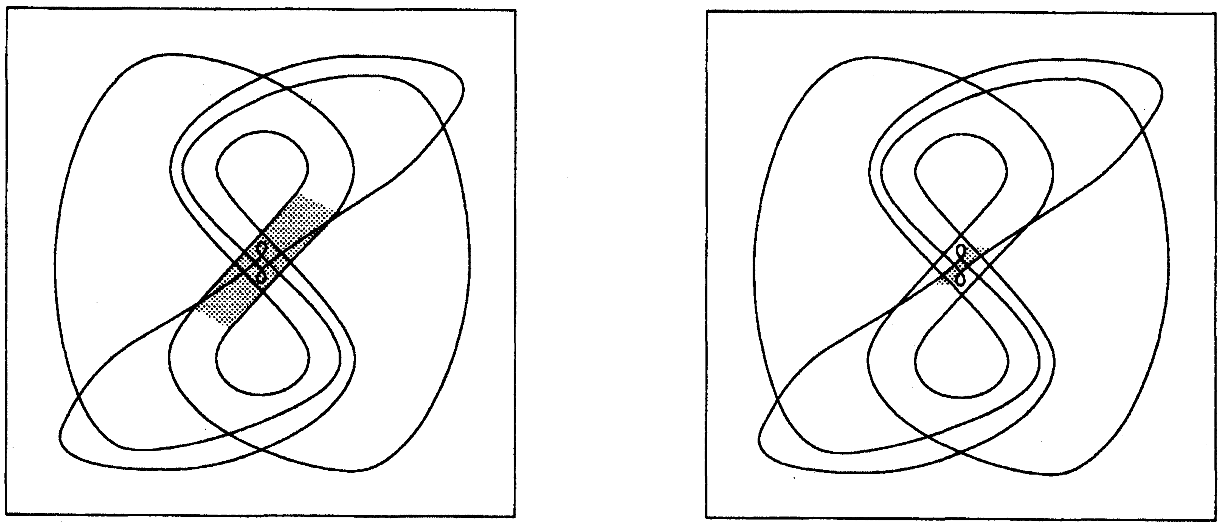
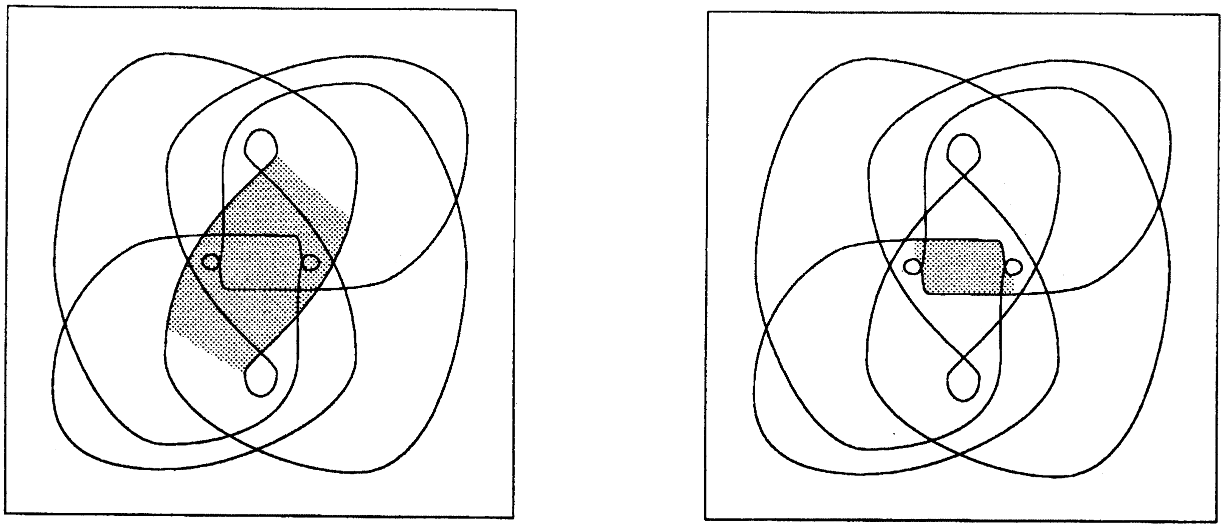
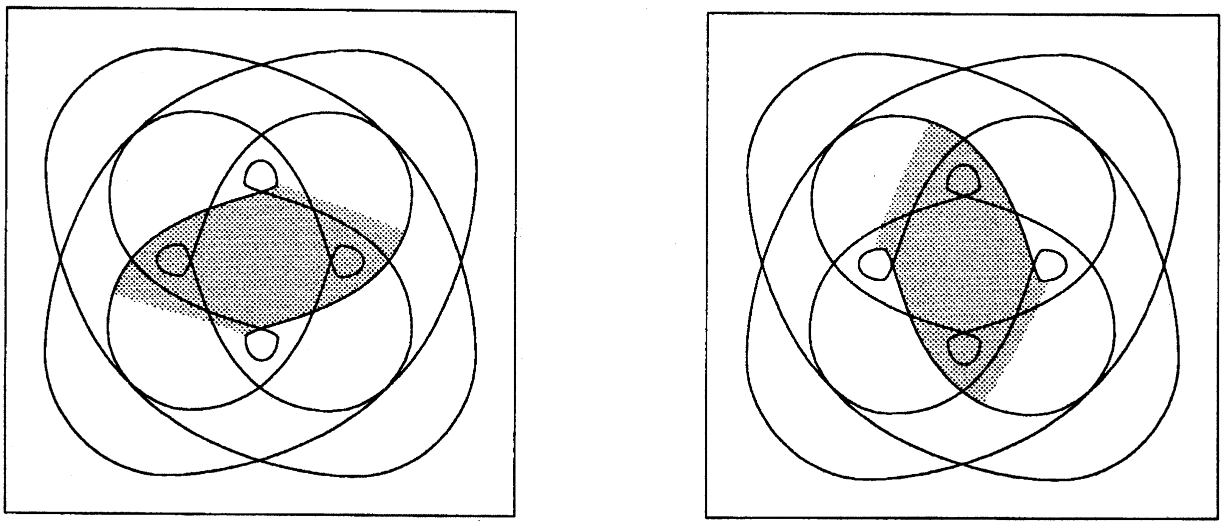


FIGURE 9. The stable fold map e_{s_3}



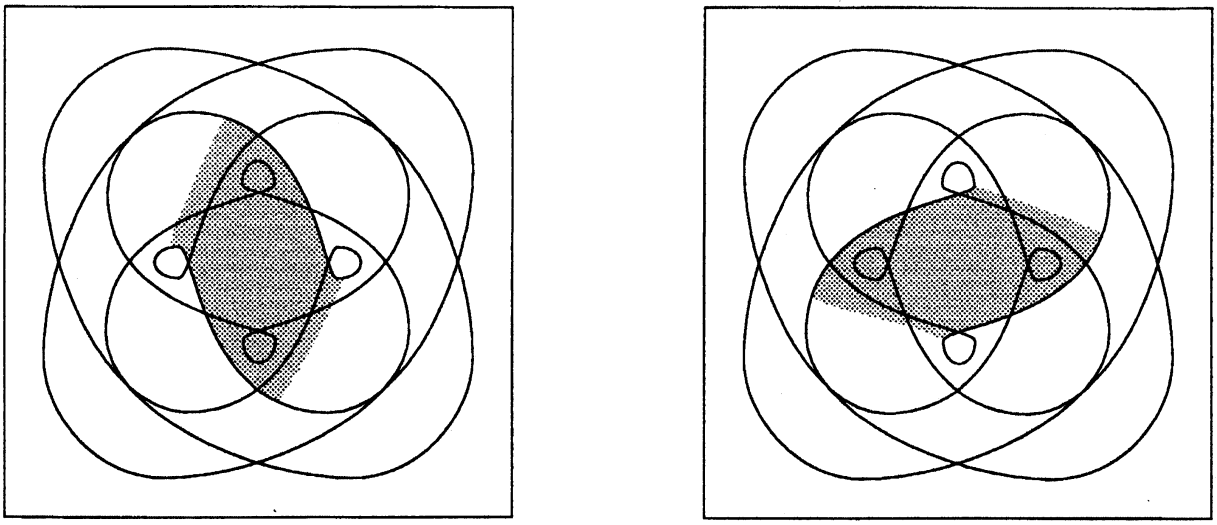
$e_{s_4}(D^2_N)$ $e_{s_4}(D^2_S)$

FIGURE 10. The stable fold map e_{s_4}



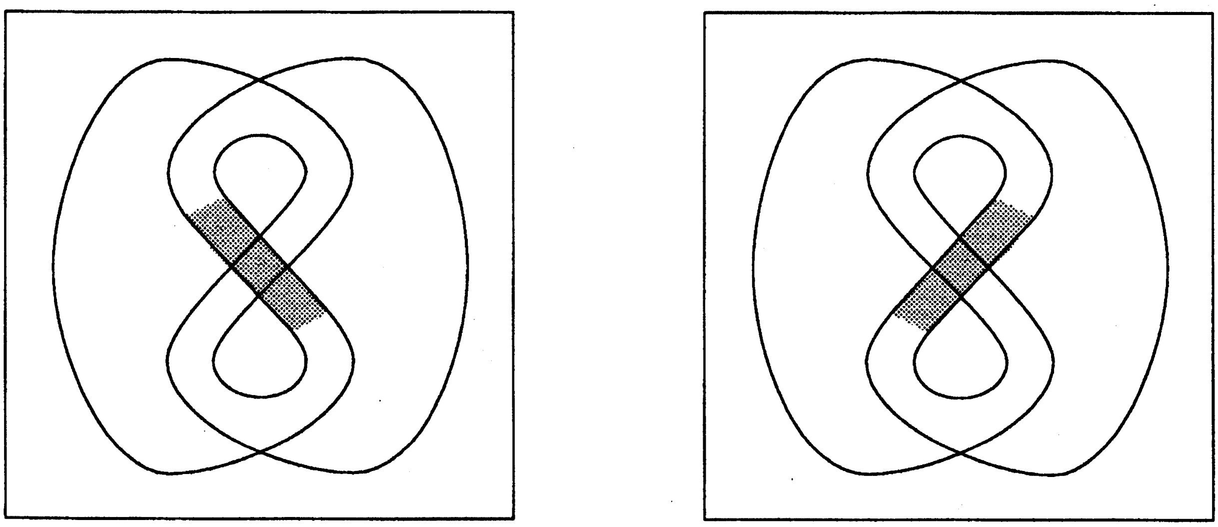
$e_{s_5}(D^2_N)$ $e_{s_5}(D^2_S)$

FIGURE 11. The stable fold map e_{s_5}



$e_{s_6}(D^2_N)$ $e_{s_6}(D^2_S)$

FIGURE 12. The stable fold map e_{s_6}



$\widetilde{e}(D^2_N)$ $\widetilde{e}(D^2_S)$

FIGURE 13. The stable fold map $\widetilde{e}$